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A METHOD FOR PIECEWISE-HOMOGENEOUS SOLUTIONS IN STATIONARY PROBLEMS OF THE THEORY OF ELASTICITY†

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The problem of the interaction of system of contacting elastic orthotropic cylinders, moving with different velocities in the direction of their generatrices, is considered. Generalized orthogonality relations are obtained for the homogeneous stationary solutions of this problem and special cases of it. One of the relations is used to solve the problem of a finite parabolic punch moving over an elastic strip by the method of piecewise-homogeneous solutions. The problem is reduced to a normal exponential-type Poincaré–Koch system. The system of piecewise-homogeneous solutions and the solution of the problem of a semi-infinite punch are constructed in quadratures by the Wiener–Hopf method. © 2000 Elsevier Science Ltd. All rights reserved.

Stationary mixed problems for a half-plane and a composite plane were investigated earlier in [1–5], and correspondence principles were also established in [6] between the integral equations for mixed problems of steady oscillations and steady motions of punches.

1. A SYSTEM OF MOVING CYLINDERS

The orthogonality of the homogeneous solutions. Suppose Cartesian rectangular systems and coordinates O_kxyz_k ($k=1, 2, \ldots, N$) are attached to N elastic orthotropic infinite cylinders $Q_k = \{x, y, z_k\}$: $(x, y) \in \Omega k, z_k \in (-\infty, +\infty)$, which have sections Ωk and which move translationally with constant velocity C_k with respect to fixed space, defined by the system of coordinates $Oxyz$. The generatrices S_k of the cylinder and the direction of motion are parallel to the Oz axis. In the region Q_k the coordinates Z_k and z are related as follows:

$$z = z_k + c_k t \tag{1.1}$$

where t is the time. The elastic characteristics and the density of the cylinders are uniform in Q_k, depend on k and are independent of Z_k and t. On the cylindrical surfaces $\Gamma_{k_1} \subset S_k$ the cylinders are in contact with one another under conditions of sliding or antisliding embedding, and the boundaries $\Gamma_{k_2} = S_k \setminus \Gamma_{k_1}$ are stress-free, clamped or are under crossed homogeneous conditions. It is assumed that the boundary conditions are independent of t at infinity.

Hence, in the system of coordinates $Oxyz$ for an infinite multilayered cylinder $Q = Q_1 \cup Q_2 \cup Q_N$ with cross-section $\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_N$ and mutually mobile parts the problem is homogeneous and stationary.

Consider the vector function $F_k(x, y, z_k, t)$, $(x, y) \in \Omega_k$, as the components of which we take three components of the displacement vector and six components of the stress tensor in Q_k, which completely satisfy the equations of the theory of elasticity. We will also consider the stationary vector function $F = F(x, y, z)$, $(x, y) \in \Omega$ with the same components in the cylinder Q.

By virtue of the fact that the solution is stationary and using relation (1.1), we have

$$\mathbf{F}_k(x, y, z_k, t) = \mathbf{F}_k(x, y, z_k + c_k t, 0) = \mathbf{F}, \quad (x, y) \in \Omega_k \tag{1.2}$$

It follows from relations (1.1) and (1.2) that

$$\mathbf{F}'_{k_t}(x, y, z_k, t) = \mathbf{F}'_{k_z}(x, y, z, 0) z'_t = c_k \mathbf{F}'_z \tag{1.3}$$

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Similarly

$$\mathbf{F}'_{k_x}(x,y,z_k,t)=\mathbf{F}'_x,\quad \mathbf{F}'_{k_y}(x,y,z_k,t)=\mathbf{F}'_y,\quad \mathbf{F}'_{k_{z_k}}(x,y,z_k,t)=\mathbf{F}'_z \tag{1.4}$$

Relations (1.3) and (1.4) show that, in the system of coordinates $Oxyz$, Cauchy's formulae and Hooke's law (the subscript k in the elasticity constants is omitted for brevity)

$$\varepsilon_x=\partial u/\partial x,\ldots,\quad \gamma_{xz}=\partial w/\partial x+\partial u/\partial z,\ldots$$
$$\sigma_x=\beta_{11}\varepsilon_x+\beta_{12}\varepsilon_y+\beta_{13}\varepsilon_z,\ldots,\quad \tau_{xz}=\beta_{55}\gamma_{xz},\quad \tau_{xy}=\beta_{66}\gamma_{xy} \tag{1.5}$$

where β_{ij} are the orthotropy coefficients $\beta_{ij}=\beta_{ji}$, $\beta_{jj}>0$, retain their form and the equations of motion take the form

$$\partial\tau_{xz}/\partial x+\partial\tau_{yz}/\partial y+\partial\sigma_z/\partial z=\rho c^2\partial^2 w/\partial z^2,\ldots \tag{1.6}$$

where $p=p_k(x,y,z)$ are the densities of the materials, $c=c_k(x,y,z)$ are the velocities of the cylinders, and $(x,y)\in\Omega k$.

We will consider the orthogonality properties of the homogeneous solutions of this problem for Q. Suppose

$$Q^\circ=\{(x,y,z):(x,y)\in\Omega,\quad z\in[a,b]\}$$

is a finite multilayer cylinder. Since $z_k=z$ at the initial instant, then, when $t=0$,

$$Q^\circ=\bigcup_{k=1}^{N}\overset{\circ}{Q}_k,\quad \overset{\circ}{Q}_k=\{(x,y,z_k):(x,y)\in\Omega_k,\ z_k\in[a,b]\}$$

Suppose $\mathbf{u}_k^m$, $\mathbf{P}_k^m\equiv\rho_k\partial\mathbf{u}_k^m/\partial t^2$ ($m=\pm1,2,\ldots$, are the displacement and inertial force vectors in Q_k, generated by the mth root p_m of the characteristic equation of the homogeneous problem for Q°, and T_k^m is the vector of the surface stresses on S_k°—the boundary of Q_k°. The components of the vectors completely satisfy the equations of the theory of elasticity in Q_k° and, consequently, Betti's reciprocity law

$$L_{mn}=L_{nm},\quad n=\pm1,\pm2,\ldots \tag{1.7}$$
$$L_{mn}=\sum_{k=1}^{N}\left[\iiint_{\overset{\circ}{Q}_k}(\mathbf{P}_k^m,\mathbf{u}_k^n)dQ+\iint_{\overset{\circ}{S}_k}(\mathbf{T}_k^m,\mathbf{u}_k^n)dS\right]$$

where L_{mn} is the work of the stresses of the mth solution on the elastic displacements of the nth solution in a finite multilayer cylinder Q° when $t=0$ and (,) is the scalar product.

By Gauss' formula and relation (1.7) we have

$$L_{mn}=\sum_{k=1}^{N}\iint_{\overset{\circ}{S}_k}[\cos\alpha\int(\mathbf{P}_k^m,\mathbf{u}_k^n)dz+(\mathbf{T}_k^m,\mathbf{u}_k^n)]dS \tag{1.8}$$

The inner integral in (1.8) is the original with respect to z of the function $P_k^m\,u_k^n$, and α is the angle between the unit vector q of the outward normal to S_k° and the unit vector r of the Oz axis.

Starting from formulae (1.7) and (1.8) and taking into account the fact that on the side surface of the cylinder $\cos\alpha=0$ we have

$$\sum_{k=1}^{N}\iint_{\Omega_k}Z_k dxdy+\sum_{k=1}^{N}\Big(\iint_{\overset{\circ}{\Gamma}_{k1}}+\iint_{\overset{\circ}{\Gamma}_{k2}}\Big)[(\mathbf{T}_k^m,\mathbf{u}_k^n)-(\mathbf{T}_k^n,\mathbf{u}_k^m)]dS=0$$
$$Z_k=\int_a^b[(\mathbf{P}_k^m,\mathbf{u}_k^n)-(\mathbf{P}_k^n,\mathbf{u}_k^m)]dz+[(\mathbf{T}_k^m,\mathbf{u}_k^n)-(\mathbf{T}_k^n,\mathbf{u}_k^m)]_{z=a}^{z=b} \tag{1.9}$$

where Γ_{kj}° ($j=1,2$) is the part of Γ_{kj} in which $z\in(a,b)$.

On Γ_{k1}° in an orthonormalized basis of the vectors q, r, s, forming a right triple, the vectors of the displacements and surface stresses have the form

$$\mathbf{u} = \{u_q, u_r, u_s\}, \quad \mathbf{T} = \{\sigma_q, \tau_{qr}, \tau_{qs}\}$$

Hence it follows that if the contacting cylinders are under conditions of sliding or antisliding embedding, we have the following equation.

$$\sum_{k=1}^{N} \iint_{\Gamma_{k1}^{\circ}} [(\mathbf{T}_k^m, \mathbf{u}_k^n) - (\mathbf{T}_k^n, \mathbf{u}_k^m)]dS = \sum_{k=1}^{N} \iint_{\Gamma_{k1}^{\circ}} (\sigma_{kq}^m u_{kq}^n - \sigma_{kq}^n u_{kq}^m)dS \tag{1.10}$$

In view of the continuity of the normal displacements and stresses at the common points of neighbouring surfaces Γ_{k1}°, the right-hand side of (1.10) is equal to zero. The integrals over Γ_{k2}°, which occur in (1.9) are also equal to zero, since the surface Γ_{k2}° is stress-free, is rigidly restrained, and is under conditions of sliding or antisliding embedding. Hence, at $t = 0$ in a stationary orthonomalized basis, connected with the system $Oxyz$, which in this case coincides with all the systems $Oxyz_k$, we have, henceforth omitting the subscript k

$$\iint_{\Omega_{xy}} Z dx dy = 0 \tag{1.11}$$

where Ω_{xy} is the orthogonal projection of the multilayer cylinder onto the coordinate plane Oxy.

Equation (1.11) also remains true in the case of complex homogeneous solutions.

In fact, suppose u is a complex vector function, which is a homogeneous solution of the problem considered. Since the coefficients in Cauchy's formulae and in Hooke's law are real, the real and imaginary parts of u will also be homogeneous solutions. Hence, the complex vector function $F_* = \{u, v, w, \sigma_x \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}\}$ can be split into two real vector functions with the same coefficients.

Separating the variables in the homogeneous solutions

$$\mathbf{u}^m = \{u^m, v^m, w^m\} = \mathbf{u}_0^m(x, y)e^{p_m z}, \quad \mathbf{T}^m = \{\tau_{xz}, \tau_{yz}, \sigma_z\} = \mathbf{T}_0^m(x, y)e^{p_m z}$$

$$\mathbf{P}^m = -\rho c^2 p_m^2 \mathbf{u}_0^m(x, y)e^{p_m z}$$

and substituting them into (1.11), we obtain after integration with respect to z

$$\iint_{\Omega_{xy}} [\rho c^2(p_m - p_n)(\mathbf{u}_0^m, \mathbf{u}_0^n) - (\mathbf{T}_0^m, \mathbf{u}_0^n) + (\mathbf{T}_0^n, \mathbf{u}_0^m)]dxdy = 0 \quad (p_n \neq -p_m)$$

Hence, noting that $p_m \mathbf{u}_0^m = \partial u^m/\partial z$ when $z = 0$, we have

$$\iint_{\Omega_{xy}} \{\rho c^2[(\partial \mathbf{u}^m/\partial z|_{z=0}, \mathbf{u}_0^n) - (\mathbf{u}_0^m, \partial \mathbf{u}^n/\partial z|_{z=0})] - (\mathbf{T}_0^m, \mathbf{u}_0^n) + (\mathbf{T}_0^n, \mathbf{u}_0^m)\}dxdy = 0 \tag{1.12}$$

Further, substituting into (1.12) the expressions for the partial derivatives when $z = 0$

$$\partial u/\partial z = \gamma_{xz} - \partial w/\partial x = \beta_{55}^{-1}\tau_{xz} - \partial w/\partial x$$

$$\partial v/\partial z = \gamma_{yz} - \partial w/\partial y = \beta_{44}^{-1}\tau_{yz} - \partial w/\partial y$$

$$\partial w/\partial z = \beta_{33}^{-1}(\sigma_z - \beta_{31}\partial u/\partial x - \beta_{32}\partial v/\partial y)$$

obtained from (1.5), were find the following relation of generalized orthogonality for the system of moving cylinders

$$\iint_{\Omega_{xy}} [(\mathbf{M}^m, \mathbf{u}^n) - (\mathbf{u}^m, \mathbf{M}^n)]dxdy = 0 \quad (p_n \neq -p_m)$$

$$\mathbf{M}^m = \{M_1^m, M_2^m, M_3^m\}$$

$$M_1^m = (\beta_{55}^{-1}\rho c^2 - 1)\tau_{xz}^m - \rho c^2 \partial w^m/\partial x, \quad M_2^m = (\beta_{44}^{-1}\rho c^2 - 1)\tau_{yz}^m - \rho c^2 \partial w^m/\partial y \tag{1.13}$$

$$M_3^m = (\beta_{33}^{-1}\rho c^2 - 1)\sigma_z^m - \beta_{33}^{-1}\rho c^2(\beta_{31}\partial u^m/\partial x + \beta_{32}\partial v^m/\partial y), \quad z = 0$$

In particular, in y, z coordinates for the problem of plane deformation of a system of contacting N orthotropic layers $z \in (-\infty, +\infty)$, $y \in (y_0, y_1) \cup (y_1, y_2), \ldots \cup (y_{N-1}, y_N)$, $y_{k-1} < y_k$ and (1.13) we have

$$\int_{y_0}^{y_N} (M_2^m v^n - w^m M_3^n - v^m M_2^n + M_3^m w^n) dy = 0 \quad (p_n \neq -p_m) \tag{1.14}$$

Here $\partial u/\partial x \equiv 0$.

Interchanging x and z and the subscripts of the orthotropy coefficients, in accordance with formulae (1.5), for one strip $x \in (-\infty, +\infty)$, $y \in (0,1)$ we obtain

$$\int_0^1 (M_1^m u^n - v^m M_2^n - u^m M_1^n + M_2^m v^n) dy = 0 \quad (p_n \neq -p_m)$$
$$M_1^m = (\beta_{11}^{-1}\rho c^2 - 1)\sigma_x^m - \beta_{11}^{-1}\beta_{12}\rho c^2 \partial v^m / \partial y \tag{1.15}$$
$$M_2^m = (\beta_{66}^{-1}\rho c^2 - 1)\tau_{xy}^m - \rho c^2 \partial u^m / \partial y$$

The orthogonality relations (1.13)–(1.15) enable one to apply the method of piecewise-continuous solutions to mixed stationary problems for systems of infinite elastic cylinders and layers.

Since the composite region Ω at any instant is mirror-symmetrical about the $z = 0$ plane, in this plane, in addition to the homogeneous solution $\mathbf{u}^m \equiv \mathbf{u}_0^m$ there is a solution $\mathbf{u}_0^q = \{u_0^m, n_0^m, -w_0^m\}$, mirror-symmetrical to it, for the region Ω, comprised of the cylinders Ω_k, moving with the previous values of the velocities but in the opposite direction $-C_k$. However, the square of the velocity c^2 occurs in the elasticity equations (1.5) and (1.6), and hence the function $\mathbf{u}_0^q$ is also a solution of the problem for Ω_1, in which the velocities have the initial direction c_k. This solution is orthogonal to the solution u_0 in the sense of (1.13). Hence, taking into account the equalities $M_1^q = -M_1^m$, $M_2^q = -M_2^m$, $M_3^q = -M_3^m$, which follow from (1.13), we have

$$\iint_{\Omega_{xy}} (-M_1^m u^n - M_2^m v^n + M_3^m w^n - u^m M_1^n - v^m M_2^n + w^m M_3^n) dx dy = 0 \quad (p_n \neq p_m) \tag{1.16}$$

Adding equalities (1.13) and (1.16), we obtain the stronger orthogonality relation

$$\iint_{\Omega_{xy}} (u^m M_1^n + v^m M_2^n - M_3^m w^n) dx dy = 0 \quad (p_n^2 \neq p_m^2) \tag{1.17}$$

Similarly, from (1.14) and (1.15) we have

$$\int_{y_0}^{y_N} (M_2^m v^n - w^m M_3^n) dy = 0, \quad \int_0^1 (M_1^m u^n - v^m M_2^n) dy = 0 \quad (p_n^2 \neq p_m^2) \tag{1.18}$$

Relations (1.17) and (1.18) are usually employed to solve boundary-value problems for finite and semi-infinite cylinders, on the ends of which crossed conditions are imposed, for example, the sliding embedding condition. Here, these problems are incompatible with the stationarity condition (1.1). However, relations (1.17) and (1.18) are also more effective than (1.13)–(1.15) for solving mixed problems for infinite cylinders which are in contact with finite rings or punches.

2. FORMULATION OF THE PROBLEM OF A FINITE PUNCH ON A STRIP

Consider the problem of the motion, with constant velocity c, of an orthotropic strip $-\infty < x_1 < +\infty$, $0 \leq y_1 \leq 1$ with respect to a symmetrical parabolic punch impressed into it. The base of the punch in a fixed system of coordinates $O_0x_0y_0$, $x_0 = x_1 + ct$, $y_0 = y_1$, is described by the equation.

$$y_0 = \alpha_0 x_0^2 - \alpha_0 l^2 - \alpha_2 + 1, \quad x_0 \in (-l, l) \tag{2.1}$$

where α_1, α_2 and l are certain positive numbers. Suppose the base of the strip $y_1 = 0$ is rigidly clamped, there is no friction between the punch and the strip, the stresses outside the punch are zero, the velocity of motion is less than the velocity of propagation of Rayleigh waves c_R in the orthotropic material, and the local deformation energy of the strip under the edge of the punch is limited.

In the half-strip $x_0 < 0$, 0, $y_0 < 1$ the solution will be sought in the form of the sum of the inhomogeneous solution of the problem of a semi-infinite punch over the whole of the strip with boundary conditions

$$u = v = 0\ (y = 0), \quad \tau_{xy} = 0\ (y = 1) \tag{2.2}$$

$$\sigma_y = 0\ (x < 0, y = 1), \quad v = \alpha_0 x^2 - 2\alpha_0 lx - \alpha_2\ (x > 0, y = 1) \tag{2.3}$$

corresponding to (2.1) in a fixed system of coordinates $Oxy, x = x_0 + l, y = y_0$ and a series of piecewise-homogeneous solutions of the same problem with singularities at $x = +\infty$. When $x_0 > 0, 0 < y_0 < l$ we will construct the solution in the system Oxy, where $x = S_0 - l$, in a similar form with fundamental conditions (2.2) and mixed conditions.

$$v = \alpha_0 x^2 + 2\alpha_0 lx - \alpha_2\ (x < 0, y = 1), \quad \sigma_y = 0\ (x > 0, y = 1) \tag{2.4}$$

with singularities in the piecewise-homogeneous solutions when $x = -\infty$. We will find the coefficients in the series in piecewise-homogeneous solutions using the orthogonality relation (1.15) from the condition of continuity of the solutions in the interval $x_0 = 0, 0 < y_0 < l$.

We will construct a general solution of the problem in the strip $-\infty < x < +\infty, 0 < y < l$. Interchanging the coordinates x and z and substituting expressions (1.5) into (1.6), we obtain equations in the displacements. Hence, using the Laplace transformation

$$u(x, y) = \frac{1}{2\pi i}\int_L U_1(p, y)e^{px}dp, \quad v(x, y) = \frac{1}{2\pi i}\int_L U_2(p, y)e^{px}dp \tag{2.5}$$

where L is the straight line $\operatorname{Re} p = \varepsilon$, we obtain

$$\begin{aligned} &(\beta_{11} - \rho c^2)p^2 U_1 + \beta_{66}U_1'' + (\beta_{12} + \beta_{66})pU_2' = 0 \\ &(\beta_{66} - \rho c^2)p^2 U_2 + \beta_{22}U_2'' + (\beta_{21} + \beta_{66})pU_1' = 0; \quad U_j'(p, y) \equiv \partial U_j / \partial y, \quad j = 1, 2 \end{aligned} \tag{2.6}$$

The solution of system (2.6) has the form

$$\begin{aligned} &U_1(p, y) = A_{1-}(B_1 \sin r_- py + B_2 \cos r_- py) + A_{1+}(B_3 \sin r_+ py + B_4 \cos r_+ py) \\ &U_2(p, y) = A_{2-}(-B_1 \cos r_- py + B_2 \sin r_- py) + A_{2+}(-B_3 \cos r_+ py + B_4 \sin r_+ py) \\ &A_{1\pm} = (\beta_{12} + \beta_{66})r_\pm, \quad A_{2\pm} = \rho c^2 - \beta_{11} + \beta_{66}r_\pm^2, \quad r_\pm = \{[\lambda_1 \pm (\lambda_1^2 - 4\lambda_0\lambda_2)^{1/2}](2\lambda_0)^{-1}\}^{1/2} \\ &\lambda_0 = \beta_{22}\beta_{66}, \quad \lambda_1 = (\beta_{11} - \rho c^2)\beta_{22} + (\beta_{66} - \rho c^2)\beta_{66} - (\beta_{12} + \beta_{66})^2, \\ &\lambda_2 = (\beta_{11} - \rho c^2)(\beta_{66} - \rho c^2) \end{aligned} \tag{2.7}$$

The quantities r_- and r_+ are positive real numbers and B_q are arbitrary functions of p; $q = 1, 2, 3, 4$.

3. SOLUTION OF THE INHOMOGENEOUS PROBLEMS OF A SEMI-INFINITE PUNCH

Consider the first problem (2.2), (2.3). Substituting (2.7) into conditions (2.2), we obtain

$$\begin{aligned} &B_1 = -A_{2-}^{-1}A_{2+}B_3, \quad B_2 = -r_-^{-1}r_+B_4 \\ &B_3 = A_{2-}(r_+E_{1-} \sin r_- p - r_- E_{1+} \sin r_+ p)C(p) \\ &B_4 = r_-(A_{2+}E_{1-} \cos r_- p - A_{2-}E_{1+} \cos r_+ p)C(p) \\ &E_{1\pm} = A_{1\pm}r_\pm - A_{2\pm} = \beta_{11} - \rho c^2 + \beta_{12}r_\pm^2 \end{aligned} \tag{3.1}$$

where $C(p)$ is an arbitrary function.

Satisfying the mixed boundary conditions (2.3), we obtain

$$\sigma^+(p)+\sigma^-(p)=N_1(p)C(p),\quad V^+(p)+V^-(p)=N_2(p)C(p) \tag{3.2}$$

$$\sigma^+(p)=\int_0^{+\infty}\sigma_y(x,1)e^{-px}dx,\quad \sigma^-(p)=\int_{-\infty}^{0}\sigma_y(x,1)e^{-px}dx=0$$

$$V^+(p)=\int_0^{+\infty}v(x,1)e^{-px}dx=\frac{2\alpha_0-2\alpha_0 lp-\alpha_2 p^2}{p^3},\quad V^-(p)=\int_{-\infty}^{0}v(x,1)e^{-px}dx$$

$$N_1(p)\equiv[\beta_{12}pU_1(p,1)+\beta_{22}U_2'(p,1)]/C(p)=p[D_1\cos s_+p+D_2\cos s_-p+D_3]$$

$$N_2(p)\equiv U_2(p,1)/C(p)=(\beta_{12}+\beta_{66})(\rho c^2-\beta_{11})s_-s_+[D_4\sin s_+p+D_5\sin s_-p]$$

$$2D_1=(\beta_{12}+\beta_{66})(\beta_{11}-\rho c^2+\beta_{66}(\lambda_2/\lambda_0)^{1/2})s_-^2R(c)$$

$$2D_2=(\beta_{12}+\beta_{66})(\beta_{11}-\rho c^2-\beta_{66}(\lambda_2/\lambda_0)^{1/2})s_+^2[R(c)+2(\beta_{11}-\rho c^2)\rho c^2]$$

$$D_3=2\beta_{12}(\beta_{12}+\beta_{66})^2(\beta_{11}-\rho c^2)(\lambda_2/\lambda_0)^{1/2}[(\beta_{11}-\rho c^2)\beta_{66}^{-1}+(\beta_{12}+\rho c^2)\beta_{22}^{-1}]$$

$$2D_4=s_-(\beta_{11}-\rho c^2+\beta_{66}(\lambda_2/\lambda_0)^{1/2}),\quad 2D_5=s_+(\rho c^2-\beta_{11}+\beta_{66}(\lambda_2/\lambda_0)^{1/2})$$

$$R(c)=[(\beta_{11}-\rho c^2)\beta_{22}-\beta_{12}^2](\lambda_2/\lambda_0)^{1/2}-(\beta_{11}-\rho c^2)\rho c^2,\quad s_\pm=r_-\pm r_+$$

Here $R(c)$ is the Rayleigh function, and the plus and minus superscripts denote that the functions are analytic in the right and left half-plane, respectively.

Eliminating the function $C(p)$ in (3.2), we obtain the Wiener–Hopf equation [7]

$$V^-(p)+\frac{2\alpha_0-2\alpha_0 lp-\alpha_2 p^2}{p^3}=K(p)\sigma^+(p),\quad K(p)=\frac{N_2(p)}{N_1(p)},\quad p\in L \tag{3.3}$$

Since $N_j(\bar{p})=-N_j(\bar{p})$ and $N_j\ (p)=N_j(p)$, the complex zeros of these functions are situated symmetrically above both coordinate axes of the complex plane and are real with respect to the imaginary axis.

We will renumber the zeros of the functions N_1 and N_2, lying in the right half-plane in the order in which their real parts increase and we will denote them by a_k and b_k respectively $k=1,2,\ldots,$ $\operatorname{Re} a_k\leqslant\operatorname{Re} a_{k+1}$, $\operatorname{Re} b_k\leqslant\operatorname{Re} b_{k+1}$, $a_{-k}=-a_k$, $b_{-k}=-b_k$. We know [8], that they are situated in a certain strip of the complex plane and their real parts are given by the formulae

$$\operatorname{Re} a_k=\frac{\pi(k+\gamma)}{s_+},\quad \operatorname{Re} b_k=\frac{\pi(k+\delta)}{s_+},\quad -2\leqslant\gamma\leqslant 2,\quad -1\leqslant\delta\leqslant 2 \tag{3.4}$$

We will assume that the functions N_1 and N_2 for any velocity $c\in(0,c_R)$ have no pure imaginary zeros, with the exception of $p=0$. For the function

$$N_1(i\beta)=i\beta[D_1\operatorname{ch}s_+\beta+D_2\operatorname{ch}s_-\beta+D_3]$$

the simplest sufficient condition for there to be no zeros is that the coefficients D_1, D_2 and D_3 should be positive, which, in the prior to the Rayleigh velocity, velocity range occurs when $\beta_{12}>0$. This inequality is satisfied for the majority of orthotropic materials [9]. For the function. N_2 the absence of imaginary zeros was proved in [10].

Returning to Eq. (3.3) we note that

$$K(0)=\frac{\beta_{66}s_-s_+(\lambda_2/\lambda_0)^{1/2}}{r_+E_{2-}-r_-E_{2+}},\quad K(i\beta)\sim\frac{A_0}{|\beta|},\quad \beta\to\pm\infty$$

$$A_0=(A_{2+}E_{1-}-A_{2-}E_{1+})[E_{1-}(A_{1+}\beta_{12}+A_{2+}\beta_{22}r_+)-E_{1+}(A_{1-}\beta_{12}+A_{2-}\beta_{22}r_-)]^{-1}$$

We will first obtain a solution of the homogeneous problem

$$V_0^-(p)=K(p)\sigma_0^+(p),\quad p\in L$$

splitting it, first of all, into two Riemann problems [11]

$$V_j^-(p) = K_j(p)\sigma_j^+(p), \quad j = 1, 2$$

Putting $K_1(p) = A_0 p^{-1}\, tg\, \pi p$, we obtain

$$\sigma_1^+(p) = \frac{\Gamma(1+p)}{\Gamma(1/2+p)}, \quad V_1^-(p) = \frac{A_0}{\sigma_1^+(-p)}$$

The function $K_2(p) = K(p)/K_1(p)$ is real on the imaginary axis and has no zeros and poles

$$K_2(0) = \frac{K(0)}{A_0\pi}, \quad K_2(i\beta) = 1 + O(e^{-2\pi|\beta|}), \quad \beta \to \pm\infty$$

Since the index of the function $K_2(p)$, $p \in L$ is zero, the solution of the second Riemann problem has the form [12]

$$\sigma_2^+(p) = \exp\left\{\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\ln K_2(t)}{t-p}dt\right\}, \quad \operatorname{Re} p > 0$$

$$\sigma_2^+(i\beta) = K_2^{-1/2}(i\beta)\exp\left\{\frac{\beta}{\pi i}\int_0^{+\infty}\frac{\ln[K_2(it)K_2^{-1}(i\beta)]}{t^2-\beta^2}dt\right\}$$

It follows from Eq. (3.3) that

$$\frac{V^-(p)}{V_0^-(p)} + \frac{2\alpha_0 - 2\alpha_0 lp - \alpha_2 p^2}{p^3 V_0^-(p)} = \frac{\sigma^+(p)}{\sigma_0^+(p)}, \quad p \in L$$

On the basis of the estimates

$$\sigma_0^+(p) = O(p^{1/2}), \quad \sigma^+(p)[\sigma_0^+(p)]^{-1} = O(p^{-1}), \quad p \to \infty$$

obtained respectively from an Abel-type theorem [7], and in view of the fact that the local deformation energy of the strip is limited in the neighbourhood of the punch edge, we obtain

$$\sigma^+(p) = \sigma_0^+(p) f_-(p)$$

$$f_-(p) = \frac{1}{pV_0^-(0)}\left\{2\alpha_0\left[\left(\frac{V_0^{-*}(0)}{V_0^-(0)} - \frac{1}{p}\right)\left(l - \frac{1}{p}\right) - \frac{V_0^{-**}(0)}{2V_0^-(0)} + \left(\frac{V_0^{-*}(0)}{V_0^-(0)}\right)^2\right] - \alpha_2\right\} \tag{3.5}$$

where $V_0^*(p) \equiv dV_0/dp$, $V_0^{**}(p) \equiv d^2V_0/dp^2$. Reverting to Eqs (3.2), we obtain

$$C(p) = \frac{V_0^-(p)}{N_2(p)} f_-(p), \quad \operatorname{Re} p < 0 \tag{3.6}$$

Substitution of expression (3.6) into (3.1) completely determines the functions (2.7) and enables formulae (2.5) to be used to find the solution of inhomogeneous problem (2.2), (2.3)

$$u_{q1}^0(x, y) = \frac{1}{2\pi i}\int_{L_1}\frac{V_0^-(p)}{N_2(p)} f_-(p) U_q(p, y) e^{px} dp, \quad q = 1, 2, 3, 4$$

Here and henceforth $u_1 = u$, $u_2 = \upsilon$, $u_3 = \tau xy$, $u_4 = \sigma x$ U_3 and U_4 are Laplace transformants of the functions u_3 and u_4, and L_1 is the contour of integration, which coincides with the imaginary axis, with the exception of the neighbourhood of the point $p = 0$, which it circumvents from the right along a semicircle of small radius.

Similarly, the solution of the second problem (2.2), (2.4) has the form

$$u_{q2}^{0}(x, y) = \frac{1}{2\pi i}\int_{L_2} \frac{V_0^+(p)}{N_2(p)} f_+(p) U_q(p, y) e^{px} dp,$$

$$V_0^+(p) = A_0 \frac{\Gamma(1/2 + p)}{\Gamma(1 + p)} \exp\left\{-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\ln K_2(t)}{t - p} dt\right\}$$

The function $f_+(p)$ differs from $f_-(p)$ by the replacement of the superscript minus by plus in the second formula of (3.5), $l + 1/p$ by $l + 1/p$ and $-\alpha_2$ by $+\alpha_2$, the signs in front of the last two terms in the square brackets are replaced by the opposite signs, and the contour L_2 circumvents the point $p = 0$ from the left.

4. SUBSYSTEMS OF PIECEWISE-HOMOGENEOUS SOLUTIONS

We will construct two subsystems of piecewise-homogeneous solutions. According to Section 2 each element of the first subsystem must satisfy homogeneous conditions (2.2) and (2.3) and have a singularity when $x = +\infty$. We will represent these elements in the form of the sum of a solution of the fundamental problem, defined by conditions (2.2) and the conditions $\upsilon(x, 1) = 0$, $-\infty < x < +\infty$, which are found from (2.7) when $p = b_k$, and the solution of the correcting mixed problem, defined by conditions (2.2) and the condition

$$\sigma_y = -N_1(b_k)e^{b_k x} \quad (x < 0, y = 1), \quad v = 0 \quad (x > 0, y = 1) \tag{4.1}$$

The right-hand side of the first condition is the stresses which occur in the kth homogeneous solution of the fundamental problem.

The mixed conditions (2.2) and (4.1), written in Laplace transformants

$$\sigma^+(p) + \frac{N_1(b_k)}{p - b_k} = N_1(p)C(p), \quad V^-(p) = N_2(p)C(p)$$

lead to Wiener–Hopf equation

$$\sigma^+(p) = \frac{V^-(p)}{K(p)} - \frac{N_1(b_k)}{p - b_k}, \quad p \in L$$

Knowing the solution of the homogeneous equation $V_0^-(p) = K(p)\sigma_0^+(p)$ (see Section 3) and using the method of finding the function $C(p)$ from Section 3, we obtain

$$C(p) = \frac{N_1(b_k)V_0^-(p)}{\sigma_0^+(b_k)N_2(p)(p - b_k)}$$

Hence, according to the principle of finding these solutions, the elements of the first subsystem of piecewise-homogeneous solutions have the form

$$u_q^k(x, y) = C_k U_q(b_k, y)e^{b_k x} + \frac{C_k N_1(b_k)}{2\pi i \sigma_0^+(b_k)} \int_{L_1} \frac{V_0^-(p)}{N_2(p)(p - b_k)} U_q(p, y) e^{px} dp \tag{4.2}$$

$q = 1, 2, 3, 4; \quad k = 1, 2, \dots$

where C_k are arbitrary constants.

The second subsystem of piecewise-homogeneous solutions with a singularity at $x = -\infty$, each element of which satisfies the homogeneous boundary conditions (2.2) and (2.4), is constructed in a similar way, where its elements differ from (4.2) by replacing $V_0^-(p)$ by $-V_0^+(p)$, the contour L_1 by L_2 and $\sigma_0^+(b_k)$ by $\sigma_0^+(b_k)$ $(k = -1, -2, \dots)$, where $\sigma_0^-(p) = A_0/V_0^+(-p)$.

5. SOLUTION OF THE PROBLEM OF A FINITE PUNCH

Following Section 2, the solution of the problem of a finite punch will be sought in the form

$$u_{q1}(x, y) = u_{q1}^0(x, y) + \sum_{k=1}^{\infty} u_q^k(x, y), \quad x < l$$
$$u_{q2}(x, y) = u_{q2}^0(x, y) + \sum_{k=-1}^{-\infty} u_q^k(x, y), \quad x > -l \tag{5.1}$$

The constants C_k are obtained from the four conditions of continuity of the solution when $x_0 = 0$

$$u_{q1}(l, y) = u_{q2}(-l, y), \quad q = 1, 2, 3, 4 \tag{5.2}$$

by replacing them with linear combinations of (5.2) with $q = 1, 2, 5, 6$, where

$$u_{5j}(x, y) = (\beta_{11}^{-1}\rho c^2 - 1)u_{4j}(x, y) - \beta_{11}^{-1}\beta_{12}\rho c^2 \partial u_{2j} / \partial y$$
$$u_{6j}(x, y) = (\beta_{66}^{-1}\rho c^2 - 1)u_{3j}(x, y) - \rho c^2 \partial u_{1j} / \partial y, \quad j = 1, 2$$

We substitute (5.1) into the new condition (5.2) and expand the contour integral in series in residues. Changing the order of summation in the double sums obtained, we obtain (U_5 and U_6 are Laplace transformants of the functions u_5 and u_6, and δ_{kq} is the Kronecker delta)

$$\sum_{k=1}^{\infty} U_q(b_k, y)\left[X_k - \sum_{n=1}^{-\infty} X_{-n} T_+(-b_n, b_k) e^{-(b_k + b_n)l} - S_+(b_k) e^{-b_k l}\right] -$$
$$- \sum_{k=-1}^{\infty} U_q(b_k, y)\left[X_k - \sum_{n=-1}^{-\infty} X_{-n} T_-(-b_n, b_k) e^{(b_k + b_n)l} - S_-(b_k) e^{b_k l}\right] = \delta_{1q} R_1(y) + \delta_{6q} R_2(y)$$

$$X_k = C_k e^{|b_k|l}, \quad T_\pm(t, \tau) = \frac{N_1(t) V_0^\pm(\tau)}{N_2'(\tau)\sigma_0^\mp(t)(\tau - t)}, \quad S_\pm(t) = \mp \frac{V_0^\pm(t) f_\pm(t)}{N_2'(t)} \tag{5.3}$$

$q = 1, 2, 5, 6$

$$R_q(y) = \frac{4\alpha_0}{V_0^-(0)} \lim_{p \to 0}\left\{[lV_0^-(0) + V_0^{-*}(0)]\frac{U_q(p, y)}{pN_2(p)} - \frac{\partial}{\partial p}\left[\frac{V_0^-(p)}{pN_2(p)} U_q(p, y) e^{pl}\right]\right\}$$

$q = 1, 3, 7; \quad U_7(p, y) \equiv \partial U_1 / \partial y$

$$R_2(y) = (\beta_{66}^{-1}\rho c^2 - 1)R_3(y) - \rho c^2 R_7(y)$$

We multiply both sides of the four equations (5.3) by $-M_1^m$ and M_2^m from (1.17) and by $U_1(b_m, y)$ and $U_2(b_m, y)$, respectively, add them and integrate the result with respect to y from 0 to 1. By virtue of the generalized orthogonality relation (1.15), in which $u = U_1$ and $\upsilon = U_2$, this leads to a normal Poincaré–Koch system with bilateral determinant

$$X_m - \sum_{n=1}^{\infty} X_{-n} T_+(-b_n, b_m) e^{-(b_m + b_n)l} = S_+(b_m) e^{-b_m l} + h_{m+}$$
$$X_{-m} - \sum_{n=1}^{\infty} X_n T_-(b_n, -b_m) e^{-(b_m + b_n)l} = S_-(-b_m) e^{-b_m l} + h_{m-}$$
$$2h_{m\pm} = \left\{\int_0^1 [\pm M_1^{\pm m} R_1(y) \mp U_2(\pm b_m, y) R_2(y)] dy\right\} \times$$
$$\times \left\{\int_0^1 [M_1^{\pm m} U_1(\pm b_m, y) - M_2^{\pm m} U_2(\pm b_m, y)] dy\right\}^{-1}$$

$m = 1, 2, \ldots$

Its matrix elements, by relation (3.4), decrease exponentially with respect to the numbers of the rows and columns.

Using the method of piecewise-homogeneous solutions considered here one can solve analytical problems with any number of punches, periodic problems, and also mixed stationary problems for systems of moving elastic strips and circular cylinders, having mutual-contact parts that are finite with respect to z.

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